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# Isoperimetric Inequalities on Curved Surfaces

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In this paper we extend the solutions of Lord Rayleigh's and St. Venant's conjectures to bounded simply connected domains on curved 2-dimensional Riemannian manifolds. The conjectures investigate the relation of the fundamental tone of a vibrating membrane with fixed boundary and the torsional rigidity of cylindrical beams to the respective areas of the membrane and the cross section of the beam. Both problems are related to the isoperimetric inequality relating the area of a bounded domain to the length of its boundary, and indeed the isoperimetric inequality will be the starting point of our work. To state our results we require some definitions.

$M$ will denote a 2-dimensional manifold with complete $C^k$, $k \geqslant 2$, Riemannian metric, $\{\Omega_1, \ldots, \Omega_m\}$ will be a collection of pairwise disjoint bounded simply connected domains in $M$, such that for each $j = 1, \ldots, m$ the boundary of $\Omega_j$, $\Gamma_j$, is a simply closed continuous, piecewise regular curve in $M$ (by regular we mean $C^k$, $k \geqslant 1$, and of maximal rank). We let $\Omega = \bigcup_{j=1}^{m} \Omega_j$, and $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ be the boundary of $\Omega$. Let $A$ denote the area of $\Omega$ and $L$ the length of $\Gamma$ with respect to the given Riemannian metric. We denote the Gauss curvature function of the Riemannian metric by $K\colon M \to \mathbb{R}$.

ISOPERIMETRIC INEQUALITY (I). *Assume Gauss' curvature function satisfies*

$$K(p) \leqslant \kappa \tag{1}$$

*for all $p \in \Omega$. Then*

$$0 \leqslant L^2 - 4\pi A + \kappa A^2. \tag{2}$$

*Furthermore, equality is achieved in* (2) *if and only if $m = 1$ and $\Omega$ is a geodesic disk with constant Gaussian curvature $\kappa$.*

We note that an immediate consequence of the isoperimetric inequality (I) is the following generalization of Carlemen's inequality [5]: Let $N$ be an $n$-dimensional, $n > 2$, Riemannian manifold all of whose Riemannian sectional curvatures are bounded above by a given constant $\kappa$. Also let $\Omega$ be a minimally

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imbedded 2-dimensional simply connected domain in $N$ with continuous, piecewise regular boundary $\Gamma$. Let $A$ denote the area of $\Omega$ and $L$ the length of $\Gamma$. Then (2) is valid, with equality in (2) if and only if $\Omega$ is a totally geodesically imbedded geodesic disk in $M$ with constant curvature $\kappa$.

Indeed it is a standard calculation that the Gaussian curvature of a minimally immersed Riemannian 2-manifold is less than or equal to the sectional curvature of the tangent 2-sections in the ambient manifold. The case of equality is likewise straightforward.

For the literature on the isoperimetric inequality we refer the reader to Schmidt's paper [14] on the isoperimetric inequality in space forms of arbitrary dimension. For more recent results which also consider Carleman's inequality we refer the reader to [9]. The isoperimetric inequality (2) was mentioned in [1, p. 514], and Karcher has given in [10] a proof of (2) under the assumption that $\Gamma$ has non-negative geodesic curvature. Our method will be based on an argument of Fiala [7], also referred to in [4]. The method of Fiala is, to us, simpler and more attractive and we refer the reader to his discussion in [7, pp. 303–332] (we summarize his results in Section 1). An advantage of our method is that whereas the Rayleigh and Venant conjectures bound the fundamental tone and reciprocal of the torsional rigidity from below (Theorem 1), the Fiala inequality can be used to extend Payne and Weinberger's inequalities [12] to give respective upper bounds (Theorem 2).

Now let $\Omega$ be a simply connected domain in $M$ with continuous piecewise regular boundary $\Gamma$. $A$ denotes the area of $\Omega$ and $L$ the length of $\Gamma$. For a given real number $\kappa$ and positive number $r$ we let $\Omega(\kappa; r)$ denote the geodesic disk of radius $r$ with constant Gaussian curvature $\kappa$. If $\kappa > 0$ we shall restrict $r$ to be less than $\pi/\kappa^{1/2}$.

We let $\Delta$ denote the Laplace–Beltrami operator of the Riemannian metric acting on functions on $M$. The fundamental tone is then the smallest positive number $\lambda(\Omega)$ (it certainly exists) for which there exists a non-trivial solution to the boundary-value problem

$$\begin{aligned} \Delta u + \lambda(\Omega) u &= 0, \\ u\,|_\Gamma &= 0. \end{aligned} \tag{3}$$

The torsional rigidity $P(\Omega)$ is obtained by first solving the boundary-value problem

$$\begin{aligned} \Delta u + 2 &= 0, \\ u\,|_\Gamma &= 0, \end{aligned} \tag{4}$$

and then setting

$$P(\Omega) = 2 \int_\Omega u\, dA,$$

where $dA$ denotes the Riemannian measure on $M$.

THEOREM 1. *Assume that inequality* (1) *is valid for all* $p \in \Omega$. *If* $\kappa > 0$ *then assume that* $A < 4\pi/\kappa$. *Now pick* $r_0 > 0$ *such that the area of* $\Omega(\kappa; r_0)$ *is equal to* $A$. *Then*

$$\lambda(\Omega(\kappa; r_0)) \leqslant \lambda(\Omega), \tag{R}$$

$$P(\Omega(\kappa; r_0)) \geqslant P(\Omega). \tag{V}$$

Equality in either of the above is achieved if and only if $\Omega$ is a geodesic disk of radius $r_0$ and constant Gaussian curvature $\kappa$.

The Rayleigh conjecture (R) was originally proved by Faber and Krahn independently in 1923–1924, and Venant's conjecture (V) by Polya in 1948. Exhaustive surveys of these results and the generated literature can be found in [11, 13]. The Rayleigh conjecture for constant curvature spaces can be found in [3, 8]. We also note that Bandle [2] has already seen that inequality (2) is enough to extend Polya and Szego's exposition [13, pp. 232–235] to surfaces with curvature bounded from above. However, neither of the above discussions considers the possibility that the respective functions realizing the fundamental tone and torsional rigidity may have critical values other than the maximum of the function. Our proof considers precisely this case.

We now turn to Payne and Weinberger's inequality. For given real numbers $\kappa$ and $r_1, r_2$ satisfying $0 \leqslant r_1 < r_2$ ($<\pi/\kappa^{1/2}$ when $\kappa > 0$) we let $\Omega(\kappa; r_1, r_2)$ denote the annulus in the space form of constant Gaussian curvature $\kappa$ formed by two concentric geodesic circles—the inner one having radius $r_1$ and the outer one having radius $r_2$. On $\Omega(\kappa; r_1, r_2)$ let $\Delta_\kappa$ denote the Laplace–Beltrami operator acting on functions on $\Omega(\kappa; r_1, r_2)$ and let $\lambda(\kappa; r_1, r_2)$ be the lowest positive number for which a non-trivial solution exists for the boundary-value problem (3) except that here we only insist that $u$ vanish on the circle of radius $r_2$. Similarly in (4) replace the boundary condition by: "$u$ vanishes on the geodesic circle of radius $r_2$"; and set

$$P(\kappa; r_1, r_2) = 2 \int_{\Omega(\kappa; r_1, r_2)} u \, dA_\kappa,$$

where $dA_\kappa$ denotes the Riemannian measure on the 2-dimensional space form of constant Gaussian curvature $\kappa$.

THEOREM 2. *Let* $\kappa \geqslant 0$ *and let* $\Omega$, $\Gamma$, $A$, $L$ *be as in Theorem* 1. *Also assume that* (1) *is valid for all* $p \in \Omega$. *If* $\kappa > 0$ *then assume* $L < 2\pi/\kappa^{1/2}$ *and* $A < 4\pi/\kappa$. *To* $\Omega$ *we associate* $\Omega(\kappa; r_1, r_2)$ *as follows: First pick* $r_2$ *so that the length of the geodesic circle of radius* $r_2$ *and of constant Gaussian curvature* $\kappa$ *is* $L$. *If* $\kappa = 0$ *then by the isoperimetric inequality* (I) *the area of* $\Omega(0; r_2)$ *is greater than or equal to* $A$. *If* $\kappa > 0$ *and the area of* $\Omega(\kappa; r_2)$ *is less than* $A$ *then exchange* $r_2$ *for* $\pi/\kappa^{1/2} - r_2$. *Thus* $r_2$ *is now chosen so that the area of* $\Omega(\kappa; r_2)$ *is greater than* $A$

*and the length of the circle of radius $r_2$ is $L$. Second, pick $r_1$ so that the area of $\Omega(\kappa; r_1, r_2)$ is equal to $A$. Then*

$$\begin{aligned} \lambda(\Omega) &\leqslant \lambda(\kappa; r_1, r_2), \\ P(\Omega) &\geqslant P(\kappa; r_1, r_2). \end{aligned} \tag{P–W}$$

The characterizations of $\lambda(\Omega)$ and $P(\Omega)$ which we shall use to prove Theorems 1 and 2 will be variational, viz., let $H$ be the space of functions which are continuous on $\bar{\Omega}$, vanish on the boundary $\Gamma$, and have square-summable gradient over $\Omega$. Then

$$\lambda(\Omega) = \min_{\substack{v \in H \\ v \neq 0}} \frac{\int_\Omega |\operatorname{grad} v|^2 \, dA}{\int_\Omega v^2 \, dA} \tag{5}$$

and

$$P(\Omega) = \max_{\substack{v \in H \\ v \neq 0}} \frac{4\{\int_\Omega v \, dA\}^2}{\int_\Omega |\operatorname{grad} v|^2 \, dA}; \tag{6}$$

cf., e.g., [13, pp. 87–91]. The expressions $|\operatorname{grad} v|$ above denote the length of the gradient of $v$ with respect to the Riemannian metric in question (when speaking later of the length in the space form of curvature $\kappa$ we will write $| \ |_\kappa$).

We note that in what follows, $d( \, , \, )$ will denote the standard distance function on $M$ which is induced by the Riemannian metric in question.

## 1. The Isoperimetric Inequality in the Real Analytic Case

In this section we first consider the case where $M$ is a real analytic 2-dimensional manifold with analytic (this and all subsequent uses of the word "analytic" mean "real analytic") Riemannian metric.

For the moment we assume that $\Omega$ is a simply connected domain in $M$ bounded by a regular analytic Jordan curve $\Gamma$. For $t > 0$ we set

$$\Omega_t = \{q \in \Omega: d(q, \Gamma) > t\},$$
$$\Gamma_t = \{q \in \Omega: d(q, \Gamma) = t\}.$$

Also, we let $A(t)$ denote the area of $\Omega_t$, $A$ the area of $\Omega$, and $L(t)$ the length of $\Gamma_t$, $L$ the length of $\Gamma$. Finally, let $T$ be the maximum distance from $\Gamma$ to its cut locus in $\Omega$, $C(\Gamma)$. We shall need the following results of [7, pp. 303–332]. First, for $t \in [0, T)$, $\Gamma_t$ is the union of at most a finite number of piecewise regular Jordan curves and points; for $t = T$, $\Gamma_t$ is a finite number of points. Second, the function $L(t)$ is continuous on $[0, T]$ and is analytic for all but at most a finite number of $t$ in $(0, T)$. Also, $A(t)$ is continuously differentiable and $A'(t) = -L(t)$ for all $t \in (0, T)$ (the prime henceforth denotes differentiation). Third, we call $t \in (0, T)$ non-singular if $C(\Gamma)$ is regular analytic all points $q \in C(\Gamma) \cap \Gamma_t$. Then the number of singular values of $t \in (0, T)$ is at most finite. Finally,

F. Fiala's Inequality. *Let $t_0 \in (0, T)$ be non-singular. Then $L(t)$ is analytic at $t_0$ and*

$$L'(t_0) \leqslant -2\pi + \int_{\Omega_{t_0}} K\, dA. \tag{1.1}$$

*Equality is obtained in* (1.1) *if and only if $\Gamma_{t_0}$ consists of one regular Jordan curve.*

We now turn to the isoperimetric inequality; for the analytic case we shall prove a stronger result which we will need for Theorem 2. We consider pairwise disjoint simply connected domains $\Omega_1, \ldots, \Omega_m$ with respective regular analytic boundaries $\Gamma_1, \ldots, \Gamma_m$. For $t > 0$ and $j = 1, \ldots, m$ set

$$\Omega_{t,j} = \{q \in \Omega_j : d(q, \Gamma_j) > t\},$$
$$\Gamma_{t,j} = \{q \in \Omega_j : d(q, \Gamma) = t\}$$

and let $A_j(t)$ denote the area of $\Omega_{t,j}$, $A_j$ the area of $\Omega_j$, $L_j(t)$ the length of $\Gamma_{t,j}$, $L_j$ the length of $\Gamma_j$, and $T_j$ the maximum distance from $\Gamma_j$ to its cut locus $C(\Gamma_j)$ in $\Omega_j$. Finally set

$$\Omega = \bigcup_{j=1}^{m} \Omega_j, \qquad \Gamma = \bigcup_{j=1}^{m} \Gamma_j, \qquad T = \max(T_1, \ldots, T_m),$$

$$L(t) = \sum_{j=1}^{m} L_j(t), \qquad A(t) = \sum_{j=1}^{m} A_j(t),$$

$$L = \sum_{j=1}^{m} L_j, \qquad A = \sum_{j=1}^{m} A_j.$$

Isoperimetric Inequality (II). *Assume* (1) *is valid on all of $\Omega$. Then*

$$\Delta(t) = L^2(t) - 4\pi A(t) + \kappa A^2(t)$$

*is a continuous decreasing function of $t$ on $[0, T]$ which goes to 0 as $t \to T$. Thus*

$$0 \leqslant L^2 - 4\pi A + \kappa A^2, \tag{1.2}$$

*and equality is achieved in* (1.2) *if and only if $m = 1$ and $\Omega$ is a geodesic disk of constant Gaussian curvature $\kappa$.*

*Proof.* We first note that for any given $t$ and $j$ the intersection of $\Gamma_{t,j}$ and $C(\Gamma_j)$ is at most a finite number of points [7, pp. 303–332]. Thus for each $j = 1,..., m$ we have $0 = L_j(T_j) = A_j(T_j)$ and therefore $0 = L_j(t) = A_j(t)$ for all $t \in [T_j, T]$. Let $m(t)$ be the number of sets in $\{\Omega_{t,1}, \Omega_{t,2},..., \Omega_{t,m}\}$ for which $A_j(t) > 0$.

From Fiala's inequality we conclude that $L(t)$ is continuous on $[0, T]$, and that for all but at most a finite number of values of $t \in (0, T)$, $L(t)$ is analytic and its derivative satisfies

$$\begin{aligned} L'(t) &= \sum_{j=1}^{m} L_j'(t) \\ &\leqslant -2\pi m(t) + \sum_{j=1}^{m} \int_{\Omega_{t,j}} K\, dA \\ &\leqslant -2\pi m(t) + \sum_{j=1}^{m} \kappa A_j(t) \\ &\leqslant -2\pi + \kappa A(t). \end{aligned} \tag{1.3}$$

Now $A(t)$ is continuously differentiable for all $t \in [0, T]$, analytic for all but at most a finite number of values of $t \in (0, T)$, and satisfies for all values of $t$, $A'(t) = -L(t)$. Thus for all but a finite number of values of $t \in (0, T)$ we have

$$L(t) L'(t) \leqslant -2\pi L(t) + \kappa A(t) L(t),$$

which implies $\Delta'(t)$ exists for such $t$ and satisfies

$$\Delta'(t) \leqslant 0.$$

Thus $\Delta(t)$ is continuous and decreasing. By our remarks at the beginning of this proof we have

$$\lim_{t \to T} \Delta(t) = 0.$$

Thus our first claim and inequality (1.2) are proven.

If one has equality in (1.2) then one has equality in all the lines of (1.3) for those $t$ for which (1.3) is valid. From the third and fourth lines we conclude that $m = 1$, and from the second and third lines we conclude (using arbitrarily small $t > 0$) that $K(p) \equiv \kappa$ on $\Omega$. From equality in the first and second lines we conclude that for $t < T$ and sufficiently close to $T$, $\Gamma_t$ consists of one regular Jordan curve. We therefore have: the cut locus of $\Gamma$ in $\Omega$ is $\Gamma_T$, and $\Gamma_T$ consists of one point [7, pp. 303–332]. Thus $\Omega$ is a geodesic disk of constant Gaussian curvature $\kappa$.

On the other hand if $\Omega$ is a geodesic disk of constant Gaussian curvature $\kappa$ then equality in (1.2) is straightforward. Thus the isoperimetric inequality (II) is proven.

We still need a stronger version than the one given for the case where the Gaussian curvature is not constant.

ISOPERIMETRIC INEQUALITY (III). *Assume the Gaussian curvature of $M$ satisfies* (1) *on all of $M$ and assume there exists a domain $D$ such that $\bar{D}$ is compact and $K(p) < \kappa$ for all $p \in \bar{D}$. Then there exists a constant $c > 0$ such that if* (a) *$\Omega$ is the union of a finite number of pairwise disjoint simply connected domains with regular analytic boundary $\Gamma$,* (b) *$\bar{D} \subset \Omega$,* (c) *$\rho = d(\Gamma, \bar{D})$ and $\Omega^\rho = \{q \in \Omega: d(q, \Gamma) < \rho\}$ then*

$$c \cdot \text{area } \Omega^\rho \leqslant L^2 - 4\pi A + \kappa A^2. \tag{1.4}$$

*Proof.* Clearly there exists $c_1 > 0$ such that $c_1 \leqslant \kappa - K(p)$ for all $p \in \bar{D}$. Set $c = 2c_1 \cdot \text{area } \bar{D}$. Then for all $t \in [0, \rho]$ we have $D \subset \Omega_t$, which implies

$$\begin{aligned} L'(t) &\leqslant -2\pi + \int_{\Omega_t} K\, dA \\ &\leqslant -(2\pi + c/2) + \kappa A(t) \end{aligned}$$

for all $t \in (0, \rho)$ for which Fiala's inequality is valid. The argument of the isoperimetric inequality (II) then implies

$$\begin{aligned} c \cdot \text{area } \Omega^\rho &= c(A - A(\rho)) \\ &\leqslant c(A - A(\rho)) + \Delta(\rho) \\ &\leqslant L^2 - 4\pi A + \kappa A^2 \end{aligned}$$

and the claim is proven.

## 2. THE ISOPERIMETRIC INEQUALITY IN THE NON-ANALYTIC CASE

$M$ is now a $C^k$, $k \geqslant 2$, 2-dimensional manifold with a complete $C^k$ Riemannian metric, whose metric tensor we denote in local coordinates by $g_{jk}$. The introduction of isothermal coordinates in $M$ gives rise to a conformal structure and, in particular, we distinguish a real analytic atlas on $M$. All local expressions and approximations will take place in this atlas. $\Omega$ is an open set in $M$ which is the union of a finite number of simply connected domains each of which is bounded by a piecewise regular Jordan curve. Let $A$ denote the area of $\Omega$, and $L$ the length of the boundary $\Gamma$. Finally we assume that $\kappa$ is the supremum of the Gaussian curvature function on $M$.

To prove the isoperimetric inequality (I) we approximate $g_{jk}$ on an open set containing $\bar{\Omega}$ by a sequence of analytic Riemannian metrics $\{g_{jk,n}: n = 1, 2,...\}$ such that in any local coordinate system we have

$$\lim_{n\to\infty} g_{jk,n} = g_{jk},$$

$$\lim_{n\to\infty} \partial_l g_{jk,n} = \partial_l g_{jk},$$

$$\lim_{n\to\infty} \partial_h \partial_l g_{jk,n} = \partial_h \partial_l g_{jk},$$

where $\partial_k$ denotes partial differentiation in local coordinates and the limits are uniform. Write $\Omega$ as $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ with boundaries $\Gamma_1,..., \Gamma_m$, respectively, each of which is given by the map $\omega_j$: $[0, 1] \to M$, where $\omega_j$ is a piecewise regular Jordan path in $M$. Approximate $\omega_j$ by a sequence $\omega_{j,n}$: $[0, 1] \to M$ of regular analytic Jordan paths in $M$ which satisfy

$$\lim_{n\to\infty} \omega_{j,n} = \omega_j$$

uniformly on $[0, 1]$, and

$$\lim_{n\to\infty} \dot{\omega}_{j,n} = \dot{\omega}_j$$

(the dot denotes differentiation with respect to the parameter on $[0, 1]$) uniformly on any closed interval in $[0, 1]$ on which $\dot{\omega}_j$ is continuous. Denote by $\Gamma_{j,n}$ the curves corresponding to $\omega_{j,n}$, by $\Omega_{j,n}$ the interior of $\Gamma_{j,n}$, by $L_n$ the sum of the lengths of $\Gamma_{j,n}$, and by $A_n$ the sum of the areas of $\Omega_{j,n}$, $j = 1,..., m$. Finally let $\kappa_n$ be the supremum of the Gaussian curvature function $K_n$ of the metric tensor $g_{jk,n}$ on the open set in question (containing $\bar{\Omega}$). Then

$$L_n \to L, \qquad A_n \to A \qquad \text{and} \qquad \kappa_n \to \kappa$$

as $n \to \infty$, from which we obtain the isoperimetric inequality (2) for $\Omega$.

It remains to consider the case of equality in (2). If $K(p) = \kappa$ for all $p \in \Omega$ then the strong isoperimetric inequality for constant curvature space forms [14] implies that $\Omega$ is a geodesic disk. If there exists $p \in \Omega$ for which $K(p) < \kappa$ then $p$ has a neighborhood $D$ such that $\bar{D} \subset \Omega$ and $K(q) < \kappa$ for all $q \in \bar{D}$. Let $A(D)$ denote the area of $D$ relative to the metric tensor $g_{jk}$ and $A_n(D)$ the area of $D$ relative to the metric tensor $g_{jk,n}$.

Now let $\rho$ denote the distance from $\Gamma$ to $\bar{D}$ relative to the metric tensor $g_{jk}$ and $\rho_n$ the distance from $\Gamma_n$ to $\bar{D}$ relative to the metric tensor $g_{jk,n}$. Finally set

$$c = \{\inf_{q\in D} \kappa - K(q)\} \cdot A(D),$$

$$c_n = \{\inf_{q\in D} \kappa_n - K_n(q)\} \cdot A_n(D).$$

Then

$$\lim_{n\to\infty} \rho_n = \rho, \qquad \lim_{n\to\infty} c_n = c.$$

We conclude then that the isoperimetric inequality (III) remains valid for a $C^k$, $k \geqslant 2$, Riemannian metric and for $\Omega$ bounded by piecewise regular $\Gamma$. Therefore if $K(p) \not\equiv \kappa$ on $\Omega$ we have strict inequality in the isoperimetric inequality (I), and the proof is completed.

## 3. The Lord Rayleigh and St. Venant Conjectures (Theorem 1)

We first note that if $u$ solves the boundary-value problem (3) then $u$ does not vanish anywhere on $\Omega$ [6, Vol. I, pp. 451–455] so by multiplying $u$ by a constant, if necessary, we may assume that $u$ is strictly positive on $\Omega$. If $u$ solves the boundary-value problem (4) then Hopf's maximum principle [6, Vol. II, pp. 320–331] implies that $u$ must be positive on $\Omega$. For either case we set $\beta = \max_\Omega u > 0$.

Before considering $u$ we consider a Morse function $v\colon \bar{\Omega} \to [0, \beta]$ in $H$, i.e., $v$ is continuous on $\bar{\Omega}$, $C^1$ and positive on $\Omega$, vanishes on $\Gamma$ the boundary of $\Omega$, and has only a finite number of critical points in $\Omega$, all of which are nondegenerate. Then there exists an $\alpha > 0$ such that the critical values $t_j$, $j = 1,..., k$, of $v$ satisfy

$$\alpha < t_1 < \cdots < t_k = \beta.$$

For any $t \in (0, \beta)$, $t \neq t_j$, $j = 1,..., k$, the level curve

$$\Gamma_t = v^{-1}(t)$$

is the disjoint union of a finite number of imbedded circles. For $j = 1,..., k$ we have that $\Gamma_{t_j}$ is the disjoint union of a finite number of immersed circles and a finite number of points.

Now for every $t \in [0, \beta]$ let

$$L(t) = \text{length of } \Gamma_t,$$
$$A(t) = \text{area of } v^{-1}([t, \beta]).$$

Then both $L(t)$ and $A(t)$ are continuous on $[0, \beta]$, and $C^1$ except at critical values corresponding to the saddle points of $v$ and possibly $t = 0$ (for critical values corresponding to the saddle points one can show that the singularity of $A'(t)$ is given by $-a \ln(t - t_j)$, where $a$ is some positive constant, and therefore $A'(t)$ is integrable).

We now introduce coordinates on $\Omega - \bigcup_{j=1}^{k} \Gamma_{t_j}$ via the level curves and gradient lines of $v$. Call $t_0 = 0$ and set

$$\Omega_j = v^{-1}((t_{j-1}, t_j)),$$
$$s_j = \tfrac{1}{2}(t_{j-1} + t_j),$$

for each $j = 1,..., k$. For each $j = 1,..., k$ we map $\Psi: \Gamma_{s_j} \times (t_{j-1}, t_j) \to \Omega_j$ in the following manner: Let $V$ be the restriction of the vector field $(\operatorname{grad} v)/|\operatorname{grad} v|^2$ to $\Omega_j$ and $\Phi_t$ its induced flow. Then set

$$\Psi(q, t) = \Phi_{t-s_j}(q)$$

for any $(q, t) \in \Gamma_{s_j} \times (t_{j-1}, t_j)$. Then $\Psi$ indeed maps $\Gamma_{s_j} \times (t_{j-1}, t_j)$ diffeomorphically onto $\Omega_j$ and satisfies:

$$v(\Psi(q, t)) = t, \qquad t \in (t_{j-1}, t_j),$$
$$|\partial\Psi/\partial t| = |\operatorname{grad} v|^{-1},$$
$$\partial\Psi/\partial t \text{ is always perpendicular to } \Gamma_t.$$

($\partial\Psi/\partial t$ is the velocity vector of the coordinate line $q =$ const, i.e., $V = \partial\Psi/\partial t$.)

As usual $dA$ denotes the Riemannian measure on $M$ and we let $dl_t$ denote the 1-dimensional measure of $\Gamma_t$ for each $t$. Thus for each $\Omega_j$ we have

$$dA(\Psi(q, t)) = |\operatorname{grad} v|^{-1}\, dl_t(q)\, dt.$$

Now set

$$p(t) = \int_{\Gamma_t} |\operatorname{grad} v|\, dl_t\,.$$

Then $p(t)$ is continuous on $[0, \beta]$ and for each $j = 1,..., k$ we have by Fubini's theorem

$$\int_{\Omega_j} |\operatorname{grad} v|^2\, dA = \int_{t_{j-1}}^{t_j} p(t)\, dt.$$

Thus

$$\int_{\Omega} |\operatorname{grad} v|^2\, dA = \int_0^{\beta} p(t)\, dt,$$

and since

$$-A'(t) = |A'(t)| = \int_{\Gamma_t} |\operatorname{grad} v|^{-1}\, dl_t$$

we have by Cauchy's inequality

$$p(t) \mid A'(t) \mid \geqslant L^2(t). \tag{3.1}$$

Recall that $\Omega(\kappa; r)$ denotes the geodesic disk of radius $r > 0$ and constant Gaussian curvature $\kappa$, and denote its area by $\mathcal{A}(r)$ and the length of its perimeter by $\mathcal{L}(r)$. Then for every $t \in [0, \beta]$ we determine $r(t)$ by the equction

$$A(t) = \mathcal{A}(r(t)).$$

The isoperimetric inequality (I) and (3.1) imply that for all $t$ for which $A(t) \in C^1$

$$p(t) \mid A'(t) \mid \geqslant \mathcal{L}^2(r(t)). \tag{3.2}$$

We also note that $r(t) \in C^1$ when $A(t) \in C^1$; thus for non-critical values of $t$ we have

$$r'(t) = A'(t)/\mathcal{L}(r(t)).$$

We now let $a(r)$ denote the inverse function of $r(t)$, $r_0 = r(0)$ (i.e., $r_0$ is chosen so that $A = \mathcal{A}(r_0)$), and associate with the function $v$ the function $w$: $\Omega(\kappa; r_0) \to [0, \beta]$ constructed as follows: Let $(r, \theta)$ denote polar coordinates on $\Omega(\kappa; r_0)$ and set

$$\begin{aligned} f(t) &= \int_0^t \frac{\{r(\tau) \mid A'(\tau) \mid\}^{1/2}}{\mathcal{L}(r(\tau))} \, d\tau, \\ w(r, \theta) &= f(a(r)). \end{aligned} \tag{3.3}$$

Then $f$ is continuous, and $C^1$ whenever $A \in C^1$. Furthermore the integrand of $f$ is always greater than or equal to 1 by (3.2).

One now easily calculates

$$\mid \operatorname{grad} w \mid_\kappa^2 = p / \mid A' \mid,$$

from which one obtains

$$\int_{\Omega(\kappa; r_0)} \mid \operatorname{grad} w \mid_\kappa^2 dA_\kappa = \int_\Omega \mid \operatorname{grad} v \mid^2 dA.$$

But

$$w(r, \theta) \geqslant a(r) \geqslant 0 \tag{3.4}$$

for all $r$, which implies

$$\int_{\Omega(\kappa; r_0)} w \, dA_\kappa \geqslant \int_\Omega v \, dA,$$

$$\int_{\Omega(\kappa; r_0)} w^2 \, dA_\kappa \geqslant \int_\Omega v^2 \, dA.$$

Therefore for our Morse function $v$ we have

$$\lambda(\Omega(\kappa; r_0)) \leqslant \frac{\int_{\Omega(\kappa;r_0)} |\operatorname{grad} w|_\kappa^2 \, dA_\kappa}{\int_{\Omega(\kappa;r_0)} w^2 \, dA_\kappa}$$

$$\leqslant \frac{\int_\Omega |\operatorname{grad} v|^2 \, dA}{\int_\Omega v^2 \, dA},$$

$$P(\Omega(\kappa; r_0)) \geqslant \frac{4\{\int_{\Omega(\kappa;r_0)} w \, dA_\kappa\}}{\int_{\Omega(\kappa;r_0)} |\operatorname{grad} w|_\kappa^2 \, dA_\kappa}$$

$$\geqslant \frac{4\{\int_\Omega v \, dA\}^2}{\int_\Omega |\operatorname{grad} v|^2 \, dA}$$

by the respective characterizations (5) and (6) of $\lambda(\Omega(\kappa; r_0))$ and $P(\Omega(\kappa; r_0))$.

As in the case of the isoperimetric inequality, we have to strengthen the estimate of the elementary case in order to deal with approximations which will yield a characterization of equality in the inequalities. To this end we note that if $\Omega$ is not a geodesic disk of constant curvature $\kappa$ then there exists a non-critical value $t^* > 0$ and $\epsilon_0 > 0$ such that (3.2) is replaced by

$$p(t) \mid A'(t)| \geqslant \mathscr{L}^2(r(t)) + \epsilon_0 \tag{3.5}$$

for all $t \in [0, t^*]$. Set

$$\epsilon_1 = \epsilon_0/\sup\{\mathscr{L}(r(t)) \colon 0 < t \leqslant t^*\},$$

$$\epsilon_2 = \epsilon_1 t^*, \qquad \epsilon_3 = \epsilon_2{}^2 A(t^*), \qquad \epsilon_4 = \epsilon_2 A(t^*).$$

Then for all $t \in [0, t^*]$ we have: the integrand of $f$ in (3.3) is greater than or equal to $1 + \epsilon_1$. Thus for all $r$ satisfying $a(r) > t^*$, (3.4) is replaced by

$$w(r, \theta) \geqslant a(r) + \epsilon_2, \qquad w^2(r, \theta) \geqslant a^2(r) + \epsilon_2{}^2, \tag{3.6}$$

which implies

$$\begin{aligned} \lambda(\Omega(\kappa; r_0)) &\leqslant \frac{\int_\Omega |\operatorname{grad} v|^2 \, dA}{\int_\Omega v^2 \, dA + \epsilon_3}, \\ P(\Omega(\kappa; r_0)) &\geqslant \frac{4\{\int_\Omega v \, dA\}^2 + \epsilon_4{}^2}{\int_\Omega |\operatorname{grad} v|^2 \, dA}. \end{aligned} \tag{3.7}$$

We now assume that the boundary of $\Omega$ is regular, and that $u \colon \bar{\Omega} \to [0, \beta]$ solves either (3) or (4). Then by Hopf's maximum principle [6, Vol. II, pp. 320–331] the normal derivative of $u$ on the boundary never vanishes. Therefore there exists $\alpha > 0$ such that $u$ has no critical values in $[0, \alpha]$. Furthermore, if $\Omega$ is not a geodesic disk of constant Gaussian curvature $\kappa$ then there exists $t^* > 0$ such that for all $t \in [0, t^*]$, $u^{-1}([t, 1])$ is not a geodesic disk of constant Gaussian curvature $\kappa$.

Let $\{v_n\}$ be a sequence of Morse functions on $\Omega$ such that

$$\begin{aligned} v_n &= u \qquad \text{on } u^{-1}([0, \alpha]), \\ \lim_{n\to\infty} v_n &= u, \\ \lim_{n\to\infty} \operatorname{grad} v_n &= \operatorname{grad} u, \end{aligned}$$

the limits being uniform on $\Omega$. By (3.5) we have

$$\lambda(\Omega(\kappa; r_0)) \leqslant \frac{\int_\Omega |\operatorname{grad} v_n|^2\, dA}{\int_\Omega v_n^2\, dA + \epsilon_3}$$

$$P(\Omega(\kappa; r_0)) \geqslant \frac{4\{\int_\Omega v_n\, dA\}^2 + \epsilon_4^2}{\int_\Omega |\operatorname{grad} v_n|^2\, dA}$$

for all $n$ and $\epsilon_3$, $\epsilon_4^2$ are independent of $n$. We therefore conclude

$$\begin{aligned} \lambda(\Omega(\kappa; r_0)) &\leqslant \frac{\int_\Omega |\operatorname{grad} u|^2\, dA}{\int_\Omega u^2\, dA + \epsilon_3} \\ &< \lambda(\Omega), \end{aligned}$$

$$\begin{aligned} P(\Omega(\kappa; r_0)) &\geqslant \frac{4\{\int_\Omega u\, dA\}^2 + \epsilon_4^2}{\int_\Omega |\operatorname{grad} u|^2\, dA} \\ &> P(\Omega), \end{aligned}$$

and Theorem 1 is proven if $\Omega$ has a regular boundary.

It remains to consider the case if the boundary of $\Omega$ is only piecewise regular, since it is possible that $u$ can have critical values arbitrarily close to 0. One easily obtains the inequalities (R) and (V) by approximating $\Omega$ by domains with regular boundary and using the continuous dependence of $\lambda(\Omega)$ and $P(\Omega)$ on the domain $\Omega$ [6, Vol. I, pp. 419–424]. If one has equality in (R) and/or (V) then the isoperimetric inequality (III) can be used to show that $\Omega$ has constant Gaussian curvature $\kappa$. To show that $\Omega$ is a geodesic disk one has to use the strong results of the constant curvature case [3] (even though [3] considers only the case $\kappa > 0$, one can easily obtain the case $\kappa \leqslant 0$).

## 4. The L. E. Payne–H. Weinberger Inequalities (Theorem 2)

We first consider the analytic case. Here the Riemannian metric is real analytic and $\Omega$ is a bounded simply connected domain with regular analytic boundary $\Gamma$, and for $t \geqslant 0$ we define

$$\begin{aligned}
\Omega_t &= \{q \in \Omega: d(q, \Gamma) > t\},\\
\Gamma_t &= \{q \in \Omega: d(q, \Gamma) = t\},\\
A(t) &= \text{area } \Omega_t, \qquad A = \text{area } \Omega,\\
L(t) &= \text{length } \Gamma_t, \qquad L = \text{length } \Gamma.
\end{aligned}$$

By the isoperimetric inequality (II) we have that for all $t > 0$

$$L^2(t) \leqslant L^2 - 4\pi(A - A(t)) + \kappa(A^2 - A^2(t)).$$

Again let $\Omega(\kappa; r)$ denote the geodesic disk of radius $r$ with constant Gaussian curvature $\kappa$, $\mathcal{O}\!\ell(r)$ its area, and $\mathscr{L}(r)$ the length of its boundary. We define $r(t)$ to be the solution of the differential equation

$$\frac{dr}{dt} = \frac{-L(t)}{\mathscr{L}(r)}$$

which satisfies

$$\mathscr{L}(r(0)) = L.$$

The initial condition on $r(t)$ implies that there exists $t_0 > 0$ such that $r(t)$ is defined and $C^1$ on $[0, t_0]$. One immediately concludes that for $t \in [0, t_0]$ we have

$$\mathcal{O}\!\ell(r(0)) - \mathcal{O}\!\ell(r(t)) = A - A(t); \tag{4.1}$$

i.e., $r(t)$ is chosen so that the annulus in the surface of constant Gaussian curvature $\kappa$ with inner circle having radius $r(t)$ and outer circle having perimeter $L$ has the same area as $\Omega - \Omega_t$.

Since $L = \mathscr{L}(r(0))$, the isoperimetric inequality (I) implies that

$$A \leqslant \mathcal{O}\!\ell(r(0)).$$

Equation (4.1) then implies that

$$A(t) \leqslant \mathcal{O}\!\ell(r(t))$$

for all $t \in [0, t_0]$, from which we obtain

$$A^2 - A^2(t) \leqslant \mathcal{A}^2(r(0)) - \mathcal{A}^2(r(t)).$$

Since $\kappa \geqslant 0$ we have

$$\begin{aligned} L^2(t) &\leqslant L^2 - 4\pi(A - A(t)) + \kappa(A^2 - A^2(t)) \\ &\leqslant L^2 - 4\pi(\mathcal{A}(r(0)) - \mathcal{A}(r(t))) + \kappa(\mathcal{A}^2(r(0)) - \mathcal{A}^2(r(t))) \\ &= \mathcal{L}^2(r(t)), \end{aligned}$$

which implies that if $T^*$ denotes the maximal distance from $\Gamma$ to its cut locus, then $r(t)$ is in $C^1$ on $[0, T^*]$. In particular,

$$-1 \leqslant r'(t) < 0$$

for all $t \in [0, T^*]$.

In the statement of the theorem, $r_2 = r(0)$, and $r_1 = r(T^*)$. Symmetry arguments imply that solutions to the boundary-value problems (3) and (4) on $\Omega(\kappa; r_1, r_2)$ depend only on the distance of points in $\Omega(\kappa; r_1, r_2)$ to the center of $\Omega(\kappa; r_2)$; therefore it will suffice to bound $\lambda(\Omega)$ and $P(\Omega)$ from above and below, respectively, by the quotient of any radial function in the annulus $\Omega(\kappa; r_1, r_2)$.

Let $\rho: [r_1, r_2] \to \mathbb{R} \in C^1$ such that $\rho(r_2) = 0$. Pull $\rho$ back to $\bar{\Omega}$, viz., let $v: \bar{\Omega} \to \mathbb{R}$ be such that for $q \in \Gamma_t$, $v(q) = \rho(r(t))$. Then $v$ is continuous on $\bar{\Omega}$, and $C^1$ on $\bar{\Omega}$ except for the cut locus of $\Gamma$. Furthermore $v$ has square-summable gradient. Indeed,

$$\begin{aligned} \int_\Omega |\operatorname{grad} v|^2\, dA &= \int_0^{T^*} (\rho'(r(t)))^2 (r'(t))^2 L(t)\, dt \\ &\leqslant \int_0^{T^*} (\rho'(r(t)))^2 L(t)\, dt \\ &= -\int_{r_2}^{r_1} (\rho'(r))^2 \mathcal{L}(r)\, dr \\ &= \int_{\Omega(\kappa; r_1, r_2)} |\operatorname{grad} \rho|_\kappa^2\, dA_\kappa. \end{aligned}$$

However, one easily has

$$\int_\Omega v\, dA = \int_{\Omega(\kappa; r_1, r_2)} \rho\, dA_\kappa,$$

$$\int_\Omega v^2\, dA = \int_{\Omega(4; r_1, r_2)} \rho^2\, dA_4.$$

We therefore conclude

$$\lambda(\Omega) \leqslant \lambda(\kappa; r_1, r_2),$$
$$P(\Omega) \geqslant P(\kappa; r_1, r_2),$$

and Theorem 2 is proven in the analytic case.

For the non-analytic case one approximates the given data by analytic data as in Section 2. We leave the details to the reader.

*Note added in proof.* Since this paper was submitted (1975), new and improved results have appeared. See the papers of R. Osserman: (i) The isoperimetric inequality, *Bull. of Amer. Math. Soc.* **84** (1978), 1182–1238 (ii) Bonnesen-style isoperimetric inequalities, *Amer. Math. Monthly* **86** (1979), 1–29.

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